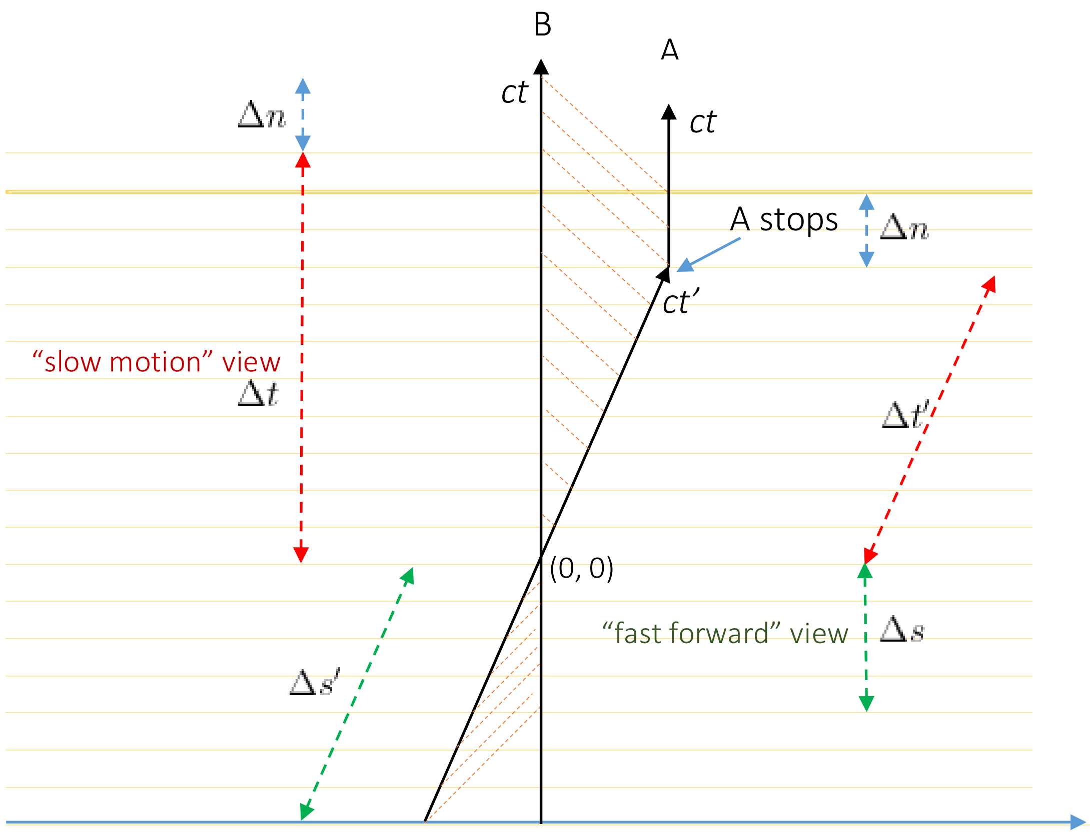
What will we see?

Space time diagrams help us to see what events are simultaneous for a given frame and help us visualize time. However, when talking about the occurrence of an event, we usually talk about the time that it occurred and not the time we could see it. There is a clear distinction between the time an event occurred and the time we see it. As a matter of fact, these two would be considered separate events.

The first event is when something happens. The second event is when light from the first event reaches an observer. Spacetime usually deals with the former type of events while astronomy is more concerned with the latter.

What is the combined effect of time dilation and light travel time which relates to what an observer will actually see i.e. the combination of temporal difference and light path travel time? We already know of frequency shifts due to motion (Doppler Effect), but here we will try to analyse the events of the observation of light. We will try to see what an observer B (stationary in space) would see from a rest frame observing light originating from a frame A in relative motion to it.

A is a rocket that moves with a velocity ***v***. A crosses B and then stops. The following diagram depicts this scenario:



Visual Doppler Effect

The orange dashed lines show the path of light from A as it reaches B. Light must travel along 45 degree lines.

For the following analysis, we will map out space time on a Cartesian plane with its origin being the point where A crosses B.

Worldline of A in motion

The equation of the worldline of A while it is in motion is given by:

$$ct = \frac{c}{v}x$$

Often, the v/c factor is taken to be β for velocity **v**. Thus, we can write:

$$\boxed{ct = \frac{x}{\beta} \qquad \text{(i)}}$$

There are three scenarios to analyse in this case. The first one is when A is moving towards B. The second one is when A is moving away from B (after it has crossed B) and the third one is when A is stationary with respect to B, but before that, we need to calculate the scale of the **ct'** axis.

Unit time on ct' axis (distance on the Cartesian plane)

The scale of time on the **ct'** axis is not the same as the scale on the **ct** axis. 1 second on the **ct'** axis corresponds to a distance on the Cartesian plane that is longer that of 1 second on the **ct** axis. The coordinate of the first 1 sec marker on the **ct'** axis (after the origin) is given by the intersection of the line

$$\boxed{ct = \gamma \qquad \text{(ii)}}$$

with the intersection A's worldline. This is because 1 second on the **ct'** axis should be γ seconds on the **ct** axis. Here γ is the Lorentz factor for the velocity **v**. Thus, combining (i) and (ii) we can obtain the position of the first 1 sec marker on the **ct'** axis in terms of the coordinates of Cartesian plane (ct, x). We equate them below:

$$\frac{x}{\beta} = \gamma$$

Therefore:

$$x = \gamma\beta$$

Hence, the point on **ct'** axis that corresponds to the 1 sec marker is given by:

$$M_1 = (\gamma\beta, \gamma)$$

Thus the distance on the **ct'** axis line (in terms of the ct, x plane) that marks 1 sec is given by the distance between $(0, 0)$ and $(\gamma\beta, \gamma)$

Thus the distance on the Cartesian plane (x, ct) that corresponds to 1 sec (along ct') is given by:

$$\Delta u = \gamma\sqrt{1+\beta^2} \qquad \text{(iii)}$$

Part I – A meets B at (0,0) and moves away from B

When A is moving away from B, the slope of a flash of light from A's frame to B's is -1. Therefore, the equation of a flash of light that is incident on the worldline of B at (0, t) on our Cartesian plane is given by:

$$ct = -x + t \quad \text{(iv)}$$

Let there be two flashes of light that are incident at t_1 and t_2. Then the equations of the flashes are:

$$ct = -x + t_1 \quad \text{(v)}$$

$$ct = -x + t_2 \quad \text{(vi)}$$

These flashes are from the frame of A therefore the point of their origin is the point of intersection of the flash equations with the worldline of B given by (i). Combining (i) and (v) we get:

$$\frac{x}{\beta} = -x + t_1$$

$$x(\frac{1}{\beta} + 1) = t_1$$

$$x = \frac{t_1}{\frac{1}{\beta} + 1}$$

Therefore:

$$ct = \frac{t_1}{\beta(\frac{1}{\beta} + 1)}$$

Therefore, origin of first flash of light F_1 is:

$$F_1 = (\frac{t_1}{(\frac{1}{\beta}+1)}, \frac{t_1}{\beta(\frac{1}{\beta}+1)}) \qquad \text{(vii)}$$

Similarly, the origin of second flash of light F_2 is:

$$F_2 = (\frac{t_2}{(\frac{1}{\beta}+1)}, \frac{t_2}{\beta(\frac{1}{\beta}+1)}) \qquad \text{(viii)}$$

The distance between F_1 and F_2 is the time expressed as distance (**ct'**) in between the two flashes of light, let's call this Δt_c (the Cartesian distance between F_1 and F_2). Then:

$$\Delta t_c^2 = (\frac{t_2 - t_1}{\beta(\frac{1}{\beta}+1)})^2 + (\frac{t_2 - t_1}{\frac{1}{\beta}+1})^2$$

We can call $t_2 - t_1$, Δs which is the time expressed as distance **ct** between the two flashes of light in B's rest frame. Then:

$$\Delta t_c^2 = (\frac{\Delta s}{\beta(\frac{1}{\beta}+1)})^2 + (\frac{\Delta s}{\frac{1}{\beta}+1})^2$$

$$\Delta t_c^2 = \frac{\Delta s^2(1+\beta^2)}{\beta^2(\frac{1}{\beta}+1)^2}$$

$$\Delta t_c = \frac{\Delta s\sqrt{1+\beta^2}}{\beta(\frac{1}{\beta}+1)} \qquad \text{(ix)}$$

This is the Cartesian distance along **ct'**. But we know from equation (iii) that the Cartesian distance divided by Δu should give us time in A's frame. Dividing both sides of (ix) with Δu, we get:

$$\frac{\Delta t_c}{\Delta u} = \frac{\Delta s\sqrt{1+\beta^2}}{\beta(\frac{1}{\beta}+1)}.\frac{1}{\gamma\sqrt{1+\beta^2}}$$

We can call $\Delta t_c/\Delta u$ as $\Delta s'$ the time in A's frame. This gives us:

$$\Delta s' = \frac{\Delta s}{\gamma\beta(\frac{1}{\beta}+1)}$$

Simplifying this we get:

$$\Delta s = \Delta s' \gamma (1+\beta) \qquad \text{(x)}$$

Since light is continuous and during the time $\Delta s'$, every instant there is a photon travelling from A's frame to B's frame and the time during which the light originated is longer than the time during which it is observed, theoretically at least, B should see light visually dilated over time by an additional factor of $(1+\beta)$ in addition to the usual time dilation factor of γ.

Thus if in A's frame there is a light that glows red for 1 second, then amber for one second and then green for one second, and B is travelling with a velocity of c/2, then:

$$\gamma = \frac{2}{\sqrt{3}}, (1+\beta) = \frac{3}{2}$$

$$\gamma(1+\beta) = \frac{3}{\sqrt{3}} = 1.73$$

Thus B should see the red light on for 1.73 sec, followed by the amber light for 1.73 sec etc. The corollary of this effect is that if we want to ignore light path differences to calculate time dilation, one can divide observed time by $(1+\beta)$.

In this case, the observed time 1.73 divided by 3/2 gives us the time dilation factor 1.15 sec. The clocks need not be brought back together to observe this (we see time visually dilated when A is moving away from B, and we can correct for this).

Part II - A moving towards B

The contents of this part are based on applying similar logic to the part I. We assume in the case of A moving towards B, the scale of time on **ct'** should remain the same. The flashes of light, however, will have a slope of 1 instead of -1.

Hence the equation of a flash of light incident at (0, t) on B's worldline is:

$$ct = x + t \qquad \text{(xi)}$$

Let there be two flashes of light that are incident at t_1 and t_2. Then the equations of the flashes are:

$$ct = x + t_1 \qquad \text{(xii)}$$

$$ct = x + t_2 \qquad \text{(xiii)}$$

These flashes are from the frame of A therefore the point of their origin is the point of intersection of the flash equations with the worldline of B given by (i). Combining (i) and (xii) we get:

$$\frac{x}{\beta} = x + t_1$$

Therefore:

$$x(\frac{1}{\beta} - 1) = t_1$$

Therefore, origin of first flash of light F_1 is:

$$F_1 = (\frac{t_1}{\frac{1}{\beta} - 1}, \frac{t_1}{\beta(\frac{1}{\beta} - 1)}) \quad \text{(xiv)}$$

Similarly, the origin of second flash of light F_2 is:

$$F_1 = (\frac{t_2}{\frac{1}{\beta} - 1}, \frac{t_2}{\beta(\frac{1}{\beta} - 1)}) \quad \text{(xv)}$$

This will lead to the equation of visual contraction of:

$$\Delta t = \Delta t' \gamma (1 - \beta) \quad \text{(xvi)}$$

Thus, in the same scenario where the light glows red for 1 second, amber for the next second etc.

$$\gamma = \frac{2}{\sqrt{3}}, (1 - \beta) = \frac{1}{2}$$

$$\gamma(1 - \beta) = \frac{1}{\sqrt{3}} = 0.5774$$

Thus B should see red light on for 0.5774 sec, followed by the amber light for 0.5774 sec etc. In actuality, the time dilation is 0.5774 divided by ½ = 2 x 0.5774 = 1.15 sec.

Part III - A and B are at rest relative to each other

In this case, we will only observe a lag which is the light travel time. Red, amber and green flashes will still be observed to be 1 sec long (this is depicted on the diagram).

Conclusion

We have used the frame invariant velocity of light (45 degree worldlines) to analyse and theorise what we should expect to see from a stationary frame in space when we observe light originating from a frame in relative motion to the stationary frame.

We have used coordinate geometry to derive equations for time as we would “see” it (as opposed to time according to when events occur). In theory, at least, time should be visually contracted when we observe light from a frame moving towards us. Also, time should be visually dilated when we observe a frame moving away from us. This has been shown by equations (x) and (xvi).